Witten Index and Superconducting Strings

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January 14, 2010

Abstract

The Yukawa interaction sector of superstring inspired models that give superconducting strings, can be described in terms of a supersymmetric quantum mechanics algebra. We relate the Witten index of susy quantum mechanics with an index characteristic to superconducting string models.

Introduction

Superconducting strings are known to have important cosmological implications [1, 2]. Cosmic strings can become superconducting if charged fermionic transverse zero modes are trapped along the strings [3]. For example in [3] a single massive fermion was considered which acquired its mass through a Yukawa-type interaction with a scalar field having varying phase around the string. In [4] an index theorem was obtained, which determines the minimum number of zero modes.

Moreover in [5] an index theorem was developed, which applies in more realistic theories. Particularly in grand-unified or superstring inspired models one has many left-right handed fermions coupled to a number of scalar fields with Yukawa interactions. Some of these models admit cosmic strings solutions and it is interesting to know which of these are superconducting. The index theorem developed in reference [5] gives an adequate solution to this and applies to models where one has a matrix of Higgs fields and many charged fermion flavors coupled to this matrix with arbitrary phase variations around the string. Moreover the nonzero index case (we denote the index I_q) is a criterion whether the cosmic strings are superconducting or not.

In this letter we shall relate the index I_q with the Witten index of supersymmetric quantum mechanical systems. Indeed we shall see that models that admit superconducting string solutions can be written in terms of a $N = 2$ supersymmetric quantum mechanics systems

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and also that the Witten index of this system is identical to the I_q index. Thus we relate a purely mathematical property of a system to the phenomenology of a grand-unified or superstring inspired model.
We shall briefly present some features of supersymmetric quantum mechanics and also the required background for superconducting strings and the index I_q, in order to make the article self contained.

Supersymmetric Quantum Mechanics and Superconducting Strings

Supersymmetric Quantum Mechanics

Let us briefly review here some properties of to supersymmetric quantum mechanics. The presentation is based on [6]. A quantum system, described by a Hamiltonian H, which is characterized by the set $\{H, Q_1, ..., Q_N\}$, with Q_i self adjoint operators, is called supersymmetric if the following anti-commutation relation holds for $i = 1, 2, ...N$,

$$\{Q_i, Q_j\} = H\delta_{i\,j} \tag{1}$$

The self-adjoint operators are then called supercharges and the Hamiltonian "H" is called SUSY Hamiltonian. The algebra (1) describes a symmetry called N-extended supersymmetry. Of course SUSY quantum mechanics can be defined in terms of non self-adjoint supercharges, as we will see shortly. The superalgebra (1) poses some restrictions on the SUSY Hamiltonian, particularly it follows due to the anti-commutation that,

$$H = 2Q_1^2 = Q_2^2 = \ldots = 2Q_N^2 = \frac{2}{N}\sum_{i=1}^{N} Q_i^2. \tag{2}$$

A supersymmetric quantum system $\{H, Q_1, ..., Q_N\}$ is said to have good susy (unbroken supersymmetry) if its ground state vanishes, that is $E_0 = 0$. For a positive ground-state energy with $E_0 > 0$, susy is said to be broken. It is obvious that for good supersymmetry, the Hilbert space eigenstates must be annihilated by all supercharges, that is,

$$Q_i|\psi_0^j\rangle = 0 \tag{3}$$

for all i, j. We now describe the basic features of $N = 2$ supersymmetric quantum mechanics. The $N = 2$ algebra consists of two supercharges Q_1 and Q_2 and a Hamiltonian H, which obey the following relations,

$$\{Q_1, Q_2\} = 0, \;\; H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2 \tag{4}$$

A more frequently used notation involves the following operators,

$$Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \tag{5}$$

and the adjoint,

$$Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \tag{6}$$

The operators of relations (5) and (6) satisfy the following equations,

$$Q^2 = {Q^\dagger}^2 = 0 \tag{7}$$

and also can be written in terms of the Hamiltonian as,

$$\{Q, Q^\dagger\} = H \tag{8}$$

It is always possible for $N = 2$ to define the Witten parity operator, W, which is defined through the following relations,

$$[W, H] = 0 \tag{9}$$

and

$$\{W, Q\} = \{W, Q^\dagger\} = 0 \tag{10}$$

Also W satisfies,

$$W^2 = 0 \tag{11}$$

Using W, we can span the Hilbert space $\mathcal{H}$ of the quantum system to positive and negative Witten-parity spaces, defined as, $\mathcal{H}^\pm = P^\pm\mathcal{H} = \{|\psi\rangle : W|\psi\rangle = \pm|\psi\rangle$. Thus the Hilbert space $\mathcal{H}$ is decomposed into the eigenspaces of W, so $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, and each operator acting on the vectors of $\mathcal{H}$ is represented in general by $2N \times 2N$ matrices. We shall use the representation

$$W = \left(\begin{array}{cc} I & 0 \\ 0 & -I \end{array}\right) \tag{12}$$

with I the $N \times N$ identity matrix. Bearing in mind that $Q^2 = 0$ and $\{Q, W\} = 0$, the supercharges are necessarily of the form,

$$Q = \left(\begin{array}{cc} 0 & A \\ 0 & 0 \end{array}\right) \tag{13}$$

and

$$Q^\dagger = \left(\begin{array}{cc} 0 & 0 \\ A^\dagger & 0 \end{array}\right) \tag{14}$$

which imply,

$$Q_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{cc} 0 & A \\ A^\dagger & 0 \end{array}\right) \tag{15}$$

and also,

$$Q_2 = \frac{i}{\sqrt{2}}\left(\begin{array}{cc} 0 & -A \\ A^\dagger & 0 \end{array}\right) \tag{16}$$

The $N \times N$ matrices A and $A^\dagger$ are generalized annihilation and creation operators. Particularly A acts as follows, $A : \mathcal{H}^- \to \mathcal{H}^+$ and $A^\dagger$ as, $A^\dagger : \mathcal{H}^+ \to \mathcal{H}^-$ In the representation

(12), (13), (14) the quantum mechanical Hamiltonian H, can be written in the diagonal form,

$$H = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \tag{17}$$

Thus for a $N = 2$ supersymmetric quantum system, the total supersymmetric Hamiltonian H, consists of two superpartner Hamiltonians,

$$H_+ = A\,A^\dagger, \quad H_- = A^\dagger\,A \tag{18}$$

The above two Hamiltonians are known to be isospectral for eigenvalues different from zero, that is,

$$\mathrm{spec}(H_+) \setminus \{0\} = \mathrm{spec}(H_-) \setminus \{0\} \tag{19}$$

The eigenstates of $P^\pm$ are called positive and negative parity eigenstates and are denoted as $|\psi^\pm\rangle$, with,

$$P^\pm|\psi^\pm\rangle = \pm|\psi^\pm\rangle \tag{20}$$

In the representation (12), the parity eigenstates are represented in the form,

$$|\psi^+\rangle = \begin{pmatrix} |\phi^+\rangle \\ 0 \end{pmatrix} \tag{21}$$

and also,

$$|\psi^-\rangle = \begin{pmatrix} 0 \\ |\phi^-\rangle \end{pmatrix} \tag{22}$$

with $|\phi^\pm\rangle \;\epsilon\; H^\pm$.

Let us now see which are the ground state properties for good supersymmetry. For good supersymmetry as we noted before, there exists at least one state in the Hilbert space with vanishing energy eigenvalue, that is $H|\psi_0\rangle = 0$. Since the Hamiltonian commutes with the supercharges, Q and $Q^\dagger$, it is obvious that, $Q|\psi_0\rangle = 0$ and $Q^\dagger|\psi_0\rangle = 0$. For a negative parity ground state,

$$|\psi_0^-\rangle = \begin{pmatrix} |\phi_0^-\rangle \\ 0 \end{pmatrix} \tag{23}$$

this implies that $A|\phi_0^-\rangle = 0$, whereas for a negative parity ground state,

$$|\psi_0^+\rangle = \begin{pmatrix} 0 \\ |\phi_0^+\rangle \end{pmatrix} \tag{24}$$

it implied that $A^\dagger|\phi_0^+\rangle = 0$. In general a ground state can have positive or negative Witten parity and when the ground state is degenerate both cases can occur. When $E \neq 0$ the number of positive parity eigenstates is equal to the negative parity eigenstates. This does not happen for the ground states. A rule to decide if there are zero modes is the so called Witten index. Let $n_\pm$ the number of zero modes of $H_\pm$ in the subspace $\mathcal{H}^\pm$. For finite n_+ and n_- the quantity,

$$\Delta = n_- - n_+ \tag{25}$$

is called the Witten index. Whenever the Witten index is non-zero integer, supersymmetry is good (unbroken). If the Witten index is zero, it is not clear whether supersymmetry is broken (which would mean $n_+ = n_- = 0$) or not ($n_+ \neq n_- \neq 0$). The Witten index is related to the Fredholm index of the operator A we mentioned earlier as,

$$\mathrm{ind}A = \dim \ker A - \dim \ker A^\dagger = \dim \ker A^\dagger A - \dim \ker A A^\dagger \tag{26}$$

The importance of the Fredholm index is that it is a topological invariant. We shall use only Fredholm operators. For a discussion on non-Fredholm operators and the Witten index, see [6]. The Witten index is obviously related to the Fredholm index of A, as,

$$\Delta = \mathrm{ind}A = \dim \ker H_- - \dim \ker H_+ \tag{27}$$

Superconducting Strings

We now briefly present the theory of superconducting strings in terms of Yukawa interactions of left-handed and right-handed fermions with Higgs scalars. We follow closely reference [5]. Consider a theory containing N left handed fermion fields ψ_α and N right-handed fermions χ_α, interacting with the Higgs sector according to the following Lagrangian,

$$\mathcal{L} = i\bar{\psi_\alpha}\gamma^\mu \partial_\mu \psi_\alpha + i\bar{\chi_\alpha}\gamma^\mu \partial_\mu \chi_\alpha - (\bar{\chi_\alpha} M_{\alpha\beta}\psi_\beta + \mathrm{H.c}). \tag{28}$$

with $\alpha, \beta = 1, ..., N$. The $N \times N$ matrix M contains the scalar fields with the interaction couplings. In general in a string background the matrix M depends only on the polar coordinates r and θ around the string. Due to the cylindrical symmetry of the string, the theory has effectively two dimensions and we can work in terms of two component spinors. The chiral fermions can be written,

$$\psi_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} \widehat{\psi_\alpha} \\ -\widehat{\psi_\alpha} \end{pmatrix} \tag{29}$$

and also,

$$\chi_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} \widehat{\chi_\alpha} \\ -\widehat{\chi_\alpha} \end{pmatrix} \tag{30}$$

Using an appropriate representation for the γ-matrices, the Lagrangian can be written,

$$\begin{aligned} \mathcal{L} &= i\widehat{\psi_\alpha}^\dagger \partial_0 \psi_\alpha - i\widehat{\psi_\alpha}^\dagger \sigma^j \partial_j \psi_\alpha i\widehat{\chi_\alpha}^\dagger \partial_0 \chi_\alpha - i\widehat{\chi_\alpha}^\dagger \sigma^j \partial_j \chi_\alpha \\ &- \widehat{\chi_\alpha^\dagger} M_{\alpha\beta} \widehat{\psi_\alpha} - \widehat{\psi_\alpha}^\dagger M_{\alpha\beta} \widehat{\chi_\alpha} \end{aligned} \tag{31}$$

The equations of motion corresponding to the Lagrangian (31) are,

$$\begin{aligned} &- \partial_0 \widehat{\psi_\alpha} + \sigma^j \partial_j \widehat{\psi_\alpha} - i M^\dagger_{\alpha\beta} \widehat{\chi_\beta} = 0 \\ &- \partial_0 \widehat{\chi_\alpha} + \sigma^j \partial_j \widehat{\chi_\alpha} - i M^\dagger_{\alpha\beta} \widehat{\psi_\beta} = 0 \end{aligned} \tag{32}$$

with $\alpha, \beta = 1, 2, ..., n$, and σ^j the Pauli matrices. Set,

$$\widehat{\psi_\alpha} = f(x_3, t) \begin{pmatrix} \psi_\alpha(r, \phi) \\ 0 \end{pmatrix} \tag{33}$$

and also,

$$\widehat{\chi_\alpha} = f(x_3, t) \begin{pmatrix} 0 \\ \chi_\alpha(r, \phi) \end{pmatrix} \tag{34}$$

Using the above two, the transverse zero-mode equations in the $x_1\, x_2$ plane, read,

$$\begin{aligned} &(\partial_1 + i\partial_2)\psi_\alpha - iM^\dagger_{\alpha\beta}\chi_\beta = 0 \\ &(\partial_1 - i\partial_2)\chi_\alpha + iM_{\alpha\beta}\psi_\beta = 0 \end{aligned} \tag{35}$$

Additionally one must have,

$$(\partial_0 - \partial_3)f = 0 \tag{36}$$

The last equation means that both ψ and χ are left movers (L-movers, see [1]). Another possibility is to have,

$$\widehat{\psi_\alpha} = f(x_3, t) \begin{pmatrix} 0 \\ \psi_\alpha(r, \phi) \end{pmatrix} \tag{37}$$

$$\widehat{\chi_\alpha} = f(x_3, t) \begin{pmatrix} \chi_\alpha(r, \phi) \\ 0 \end{pmatrix} \tag{38}$$

with corresponding equations of motion,

$$\begin{aligned} &(\partial_1 - i\partial_2)\psi_\alpha - iM^\dagger_{\alpha\beta}\chi_\beta = 0 \\ &(\partial_1 + i\partial_2)\chi_\alpha + iM_{\alpha\beta}\psi_\beta = 0 \end{aligned} \tag{39}$$

and also,

$$(\partial_0 + \partial_3)f = 0 \tag{40}$$

In this case both ψ and χ are right movers (R-movers, see [1]). The main interest in these theories is focused on the above zero modes. For a general mass matrix $M_{\alpha\beta}$, the solutions of (35) and (39) are difficult to find. We define,

$$\mathcal{D} = \begin{pmatrix} \partial_1 + i\partial_2 & -iM^\dagger \\ iM & \partial_1 - i\partial_2 \end{pmatrix}_{2N\times 2N} \tag{41}$$

and additionally,

$$\mathcal{D}^\dagger = \begin{pmatrix} \partial_1 - i\partial_2 & -iM^\dagger \\ iM & \partial_1 + i\partial_2 \end{pmatrix}_{2N\times 2N} \tag{42}$$

acting on

$$\begin{pmatrix} \psi_\alpha \\ \chi_\alpha \end{pmatrix} \tag{43}$$

The solutions of (35) and (39) are the zero modes of D and also $D^{\dagger}$. The Fredholm index I_q of the operator $\mathcal{D}^{\dagger}$, is equal to,

$$\mathrm{ind}D = \mathrm{I}_q = \dim\ker(D^{\dagger}) - \dim\ker(D) \tag{44}$$

which is the number of zero modes of $\mathcal{D}$ minus the number of zero modes of $\mathcal{D}^{\dagger}$ and equals to the number of the right movers R minus the number of the left movers L. The mass matrix is assumed to have the following form,

$$M_{\alpha\beta}(r,\phi) = S_{\alpha\beta}(r)e^{iq_{\alpha\beta}\phi} \tag{45}$$

The integers $q_{\alpha\beta}$ are related to the charges of the fields with respect to the group generator Q which corresponds to the string [1]. With $\bar{q}_{\alpha}$ and q_{β} the charges of the fermion fields $\chi_{\alpha}^{\dagger}$ and ψ_{β}, the neutrality of $\chi_{\alpha}^{\dagger}M_{\alpha\beta}\psi_{\beta}$ implies

$$q_{\alpha\beta} = \bar{q}_{\alpha} - q_{\beta} \tag{46}$$

It is proved that $I_q = \sum_{\alpha=1}^{n} q_{\alpha\alpha}$ [4, 5]. Therefore the Fredholm index of D is related to the charges of the fermions to the string gauge group.
We can see that the theory of superconducting string zero modes, defines a $N=2$ supersymmetric quantum mechanical system. Indeed we can write,

$$Q = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \tag{47}$$

and additionally,

$$Q^{\dagger} = \begin{pmatrix} 0 & 0 \\ D^{\dagger} & 0 \end{pmatrix} \tag{48}$$

Also the Hamiltonian of the system can be written,

$$H = \begin{pmatrix} DD^{\dagger} & 0 \\ 0 & D^{\dagger}D \end{pmatrix} \tag{49}$$

It is obvious that the above matrices obey, $\{Q,Q^{\dagger}\} = H$, $Q^2 = 0$, ${Q^{\dagger}}^2 = 0$, $\{Q,W\} = 0$, $W^2 = I$ and $[W,H] = 0$. Thus we can relate the Witten index of the $N=2$ supersymmetric quantum mechanics system with the index I_q of the charges that the fermions have. Indeed we have $I_q = -\Delta$, because,

$$I_q = \dim\ker D^{\dagger} - \dim\ker D = \dim\ker DD^{\dagger} - \dim\ker D^{\dagger}D = -\mathrm{ind}D = -\Delta = n_{-} - n_{+} \tag{50}$$

So it is clear that the underlying supersymmetric algebra is related to the phenomenology of the model on which the superconducting string is based. This is very valuable because one can answer the question if a model gives superconducting string solution by examining the Witten index of the corresponding $N=2$ supersymmetric algebra. Before proceeding to some examples, let us discuss some important issues. Due to the supersymmetric quantum mechanical structure of the system, the zero modes of the operator D are related

to the zero modes of the operator $DD^\dagger$. Therefore we can say that the zero modes of $DD^\dagger$ and $D^\dagger D$ can be classified according to the Witten parity, to parity positive and parity negative solutions. The last is valuable in order to find solution to the equations (35) and (39). It is known [5] that when $I_q \neq 0$ then string superconductivity is guaranteed. According to relation (50), string superconductivity occurs when the Witten index Δ is non-zero (susy unbroken). So when supersymmetry is not broken, the theory we described admits superconducting solutions. Also when the theory admits superconducting solutions (R-movers and L-movers) supersymmetry is good-unbroken. However according to [5] when I_q it is not sure whether superconducting strings exist or not. Actually there may be some cases in which solutions exist, while $I_q = 0$. Does this means that the number of R-movers is equal to L-movers or there are no zero modes? It is found in [5] that when someone uses the index I_q it is a good criterion to deal with these problems. Therefore we can decide if supersymmetry is broken or not.
Let us give an example at this point (we follow [5]). Consider a superstring inspired model based on a subgroup G of E_6, which has an additional $U(1)$ factor along with the Standard Model group, that is $G = SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_{L-R}$. The breaking of $U(1)_{L-R}$ gives rise to cosmic strings. These models contain singlets under $SU(3)_c \times SU(2)_L \times U(1)_Y$, which are responsible for the breaking of the additional $U(1)_{L-R}$. When one performs a non-trivial $L - R$ transformation, the $SU(3)_c \times SU(2)_L \times U(1)_Y$ singlets acquire a phase around the string. These fields are, $S_i = \langle S_i \rangle e^{i\phi}$, $\tilde{S}_i = \langle S_i \rangle e^{i\phi}$ $(i = 1, 2, 3)$, $N = \langle N \rangle e^{i\phi}$. The field $\tilde{S}_i$ is the mirror of S_i. The field N does not have a mirror. The model contains the Higgs doublets, $H = \langle H \rangle e^{in\phi}$ and also $\tilde{H} = \langle \tilde{H} \rangle e^{i(n+1)\phi}$, n =integer. If we examine the down quark mass matrix, ignoring fermion states from incomplete multiplets, the mass matrix is,

$$M = \begin{array}{cc} & \begin{array}{cc} g & Q_D \end{array} \\ \begin{array}{c} g_c \\ D_c \end{array} & \left(\begin{array}{c|c} \langle S_1 \rangle & 0 \\ \hline \langle N \rangle e^{i\phi} & \langle H \rangle e^{in\phi} \end{array} \right)_{18 \times 18} \end{array} \tag{51}$$

The interactions that give rise to the above mass matrix are, $g g_c S$, $g D_c N$ and $Q_D D_c N$. The heavy quark states g, and g_c mix with the d-quark states D_c and Q_D. The fermion families are 3 and each flavor has 3 colors so each block has 9×9 dimension. In the above when $n \neq -1$, then $I_q \neq 0$ (and actually $I_q = 1 + n$ according to $I_q = \sum_{\alpha=1}^{n} q_{\alpha\alpha}$) the cosmic strings are superconducting. Thus in this case the scalar-fermion sector has Witten index $\Delta \neq 0$. Therefore the quantum mechanical supersymmetry is unbroken (good supersymmetry). However when $n = -1$, then $I_q = 0$, nevertheless according to [5], there are 9 L-movers and 9 R-movers. So someone could say that supersymmetry is good (unbroken), and so the positive parity states are equal to negative parity states.

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